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# The Continuous Galerkin Method Is Locally Conservative

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We examine the conservation law structure of the continuous Galerkin method. We employ the scalar, advection–diffusion equation as a model problem for this purpose, but our results are quite general and apply to time-dependent, nonlinear systems as well. In addition to global conservation laws, we establish local conservation laws which pertain to subdomains consisting of a union of elements as well as individual elements. These results are somewhat surprising and contradict the widely held opinion that the continuous Galerkin method is not locally conservative. 

## 1. INTRODUCTION

In comparisons of discontinuous and continuous Galerkin methods, the local conservation property of the former is often identified as an advantageous property, although the precise advantage is not often explained. Let us take the point of view here that local conservation is at least desirable, possibly helpful, and certainly not harmful. Local conservation, and in particular element conservation, emanates from the property that the weighting function can be set exactly to value 1 on the subdomain or element of interest and zero elsewhere. Due to the discontinuous nature of the weighting function space, this is possible in the discontinuous Galerkin method on an element-by-element basis. (In the finite volume method, a similar property holds for the volumes, or covolumes, depending on whether the method is cell, or node, centered, respectively.)

In contrast, it is usually said that the continuous Galerkin method is globally conservative, but not locally conservative. We have trouble with this statement on both counts and are of the opinion that the conservation law structure of the continuous Galerkin method is not very well understood. Our goal in this work is to shed some light on this subject.

We begin in Section 2 by introducing a model problem which serves as a vehicle for discussing conservation. We use the steady, scalar advection–diffusion equation for this



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purpose. We also derive results for the limiting case of no diffusion, the so-called (hyperbolic) "reduced problem." Although the problem we treat is a simple one, the ideas are more general and apply to typical situations such as the compressible and incompressible Navier–Stokes equations. Likewise, the results obtained may also be generalized to the unsteady case by employing the time-discontinuous Galerkin method on space–time slabs wherein the finite element spaces are continuous within each slab.

We allow for different types of boundary conditions such as Dirichlet conditions and Neumann conditions on the total and/or diffusive flux. We distinguish between boundary conditions on inflow and outflow partitions of the body, and release outflow conditions for the reduced problem. We treat boundary conditions in the typical way; namely, Dirichlet conditions are enforced strongly, whereas Neumann conditions are enforced weakly. In each case we identify the correct conservation law structure for the theory and then proceed to investigate the same for the continuous Galerkin method. We note that all results obtained hold exactly for both Galerkin and stabilized Galerkin methods (e.g., SUPG, GLS).

We first explore global conservation in Section 3. We note that the global conservation law requires that the weighting function whose value is precisely 1 throughout the domain of the boundary value problem be present in the weighting function space. This is only the case for no Dirichlet boundary conditions, because strong enforcement of the Dirichlet condition necessitates that weighting functions take value zero on the Dirichlet portion of the boundary. Consequently, global conservation only occurs when we have all Neumann boundary conditions. In cases where there are Dirichlet conditions, we can say nothing about global conservation.

However, there is a well-known remedy to the problem of global conservation (see, e.g., Wheeler [11], Douglas *et al.* [8], Carey *et al.* [3, 4], Oshima *et al.* [10], Mizukami [9], Gresho *et al.* [5], Barrett and Elliott [2], Hughes [6, p. 107], Hughes *et al.* [7]): Introduce a modified (i.e., "mixed") formulation with an auxiliary field which amounts to the flux on the Dirichlet portion of the boundary. The modified formulation reduces to the usual continuous Galerkin method plus a "postprocessing" calculation to determine the flux. This field is expanded in terms of the basis functions omitted to satisfy the homogeneous Dirichlet boundary condition. The resulting flux possesses remarkable properties: (i) It is the missing link in the global conservation structure of the method, and (ii) it achieves superior convergence characteristics (i.e., "superconvergence," Babuška and Miller [1].) The global conservation law of the governing theory is then obtained for the (modified) continuous Galerkin method. This result then confirms the usual assertion that the continuous Galerkin method is globally conservative.

In Section 4 we examine the issue of local conservation of the continuous Galerkin method. Specifically, we endeavor to obtain a conservation law for a subdomain consisting of a union of connected element domains. It is usually thought that this is not possible because the weighting function taking on value 1 on the subdomain, and identically zero elsewhere, is not available in the continuous Galerkin method. However, we point out that the method of establishing global conservation is a paradigm capable of exposing the local conservation structure of the continuous Galerkin method as well. For the subdomain under consideration, we introduce an auxiliary boundary flux field and develop a modified formulation which reduces to the usual continuous Galerkin method plus the previous modification to attain global conservation. With the usual solution of the global auxiliary boundary flux in hand, the new modification entails a subsequent "postprocessing" calculation for the auxiliary boundary flux on the subdomain. We show that this flux is the missing link to conservation on the subdomain and show that the formulation thereby attains the exact conservation law on the subdomain. Furthermore, we show that if a similar calculation is performed with respect to the complementary subdomain, a pointwise identical balancing flux is obtained. In other words, uniqueness is achieved on the interface between the subdomains.

In Section 5 we specialize the results of Section 4 to an individual element subdomain and determine the element conservation law. This result seems to us to refute the notion that the continuous Galerkin method is not locally conservative. We also argue that the auxiliary flux is a continuous redistribution of the element nodal fluxes which likewise are a conserved quantity. In fact, all conservation properties of the auxiliary fields emanate from the conservation of nodal fluxes. This is where the fundamental conservation structure of the continuous Galerkin method resides and this is why one is able to redistribute the fluxes continuously in a conservative way. It seems that this observation has been missed heretofore. In conclusion, perhaps a more accurate characterization of the conservation comparison between discontinuous and continuous Galerkin methods is that the discontinuous Galerkin method's fundamental local conservation property is with reference to element subdomains, whereas for the continuous Galerkin method, it is with reference to nodal resultant fluxes. In the former case the conservation structure is transparent, whereas in the latter it requires elucidation through the introduction of auxiliary fluxes.

In Section 6 we present some numerical calculations in support of the theory.

The comparison of continuous and discontinuous Galerkin methods involves many aspects. We conjecture that each method will find situations in which it is preferable for various reasons. We hope that with respect to the conservation properties we have clarified and stimulated the debate.

## 2. THE SCALAR STEADY ADVECTION DIFFUSION EQUATION

### 2.1. *Preliminaries*

Let $\Omega$ be an open, bounded region in $\mathbb{R}^d$, where $d$ is the number of space dimensions, and let $\Gamma = \partial\Omega$ denote the boundary of $\Omega$, assumed piecewise smooth. The unit outward normal vector to $\Gamma$ is denoted by $\boldsymbol{n} = (n_1, n_2, \ldots, n_d)$. Let $\boldsymbol{a}$ denote the given flow velocity, assumed solenoidal, i.e., $\boldsymbol{\nabla} \cdot \boldsymbol{a} = 0$. The following notations are useful:

$$a_n = \boldsymbol{n} \cdot \boldsymbol{a}, \tag{1}$$

$$a_n^+ = (a_n + |a_n|)/2, \tag{2}$$

$$a_n^- = (a_n - |a_n|)/2. \tag{3}$$

Let $\{\Gamma^-, \Gamma^+\}$ and $\{\Gamma_{\mathrm{g}}, \Gamma_{\mathrm{h}}\}$ be partitions of $\Gamma$, where

$$\Gamma^- = \{\boldsymbol{x} \in \Gamma \mid a_n(\boldsymbol{x}) < 0\} \quad \text{(inflow boundary)}, \tag{4}$$

$$\Gamma^+ = \Gamma - \Gamma^- \quad \text{(outflow boundary)}. \tag{5}$$

Observe from (5) that we use a minus sign to denote set subtraction. The following subsets are also required (see Fig. 1)

$$\Gamma_{\mathrm{g}}^\pm = \Gamma_{\mathrm{g}} \cap \Gamma^\pm, \tag{6}$$

$$\Gamma_{\mathrm{h}}^\pm = \Gamma_{\mathrm{h}} \cap \Gamma^\pm. \tag{7}$$

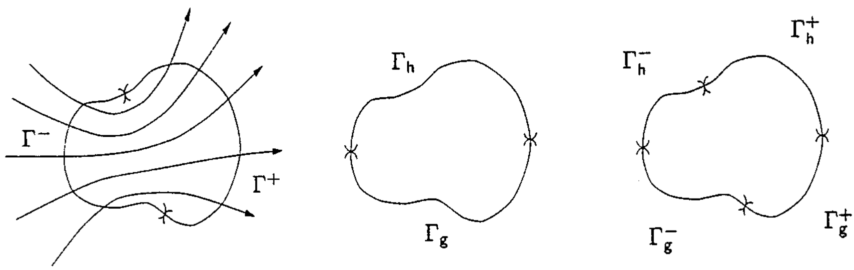



**FIG. 1.** Illustration of boundary partitions for the elliptic case.

Let $\kappa = \text{const.} > 0$ denote the diffusivity. Various fluxes are important:

$$\boldsymbol{\sigma}^{\mathrm{a}}(u) = -\boldsymbol{a}\,u \qquad \text{(advective flux)}, \tag{8}$$

$$\boldsymbol{\sigma}^{\mathrm{d}}(u) = \kappa \boldsymbol{\nabla} u \qquad \text{(diffusive flux)}, \tag{9}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{a}} + \boldsymbol{\sigma}^{\mathrm{d}} \qquad \text{(total flux)}, \tag{10}$$

$$\sigma_n^{\mathrm{a}} = \boldsymbol{n} \cdot \boldsymbol{\sigma}^{\mathrm{a}}, \tag{11}$$

$$\sigma_n^{\mathrm{d}} = \boldsymbol{n} \cdot \boldsymbol{\sigma}^{\mathrm{d}}, \tag{12}$$

$$\sigma_n = \boldsymbol{n} \cdot \boldsymbol{\sigma}. \tag{13}$$

Let $D$ denote a domain (e.g., $\Omega, \Gamma$). The $L_2(D)$ inner product and norm are denoted by $(\cdot\,,\cdot)_D$ and $\|\cdot\|_D$, respectively.

### 2.2. *Elliptic Case*

The problem consists of finding $u = u(\boldsymbol{x})\ \forall\, \boldsymbol{x} \in \overline{\Omega}$, such that

$$\mathcal{L}u \equiv -\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}(u) = \mathsf{f} \quad \text{in } \Omega, \tag{14}$$

$$u = \mathsf{g} \quad \text{on } \Gamma_{\mathsf{g}}, \tag{15}$$

$$-a_n^- u + \sigma_n^{\mathrm{d}}(u) = \mathsf{h} \quad \text{on } \Gamma_{\mathsf{h}}, \tag{16}$$

where $\mathsf{f} : \Omega \to \mathbb{R}$, $\mathsf{g} : \Gamma_{\mathsf{g}} \to \mathbb{R}$ and $\mathsf{h} : \Gamma_{\mathsf{h}} \to \mathbb{R}$ are prescribed data. The boundary condition can be understood by letting

$$\mathsf{h} = \mathsf{h}^- \quad \text{on } \Gamma_{\mathsf{h}}^- \text{ (total flux)}, \tag{17}$$

$$\mathsf{h} = \mathsf{h}^+ \quad \text{on } \Gamma_{\mathsf{h}}^+ \text{ (diffusive flux)}. \tag{18}$$

### 2.3. *Variational Formulation*

The variational form of the boundary value problem is stated in terms of the following function spaces:

$$\mathcal{S} = \{u \in H^1(\Omega) \mid u = \mathsf{g} \text{ on } \Gamma_{\mathsf{g}}\}, \tag{19}$$

$$\mathcal{V} = \{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_{\mathsf{g}}\}. \tag{20}$$

The objective is to find $u \in \mathcal{S}$, such that

$$B(w, u) = L(w) \quad \forall\, w \in \mathcal{V}, \tag{21}$$

where

$$B(w, u) \equiv (\nabla w, \boldsymbol{\sigma}(u))_\Omega + (w, a_n^+ u)_{\Gamma_\mathrm{h}}, \tag{22}$$

$$L(w) \equiv (w, \mathrm{f})_\Omega + (w, \mathrm{h})_{\Gamma_\mathrm{h}}. \tag{23}$$

The *formal consistency* of Eq. (21) with the strong form of the problem, Eqs. (14)–(16), may be verified as follows:

$$\begin{aligned} 0 &= B(w, u) - L(w) \\ &= -(w, \nabla \cdot \boldsymbol{\sigma}(u))_\Omega + (w, \sigma_n(u))_{\Gamma_\mathrm{h}} + (w, a_n^+ u)_{\Gamma_\mathrm{h}} - (w, \mathrm{f})_\Omega - (w, \mathrm{h})_{\Gamma_\mathrm{h}} \\ &= -(w, \nabla \cdot \boldsymbol{\sigma}(u) + \mathrm{f})_\Omega + (w, -a_n^- u + \sigma_n^\mathrm{d}(u) - \mathrm{h})_{\Gamma_\mathrm{h}} \quad \forall\, w \in \mathcal{V}. \end{aligned} \tag{24}$$

*Stability*, or *coercivity*, is established as follows:

$$\begin{aligned} B(w, w) &= (\nabla w, -\boldsymbol{a} w + \kappa \nabla w)_\Omega + (w, a_n^+ w)_{\Gamma_\mathrm{h}} \\ &= -\frac{1}{2}(w, a_n w)_{\Gamma_\mathrm{h}} + \kappa \|\nabla w\|_\Omega^2 + (w, a_n^+ w)_{\Gamma_\mathrm{h}} \\ &= \kappa\, \|\nabla w\|_\Omega^2 + \frac{1}{2}\left\| |a_n|^{1/2} w \right\|_{\Gamma_\mathrm{h}}^2 \quad \forall\, w \in \mathcal{V}. \end{aligned} \tag{25}$$

## 2.4. *Hyperbolic Case ("Reduced Problem")*

In the absence of diffusion we cannot specify a boundary condition on the outflow boundary. This time we employ the partition $\Gamma = \Gamma_\mathrm{g}^- \cup \Gamma_\mathrm{h}^- \cup \Gamma^+$ and we define $\Gamma_\mathrm{g} = \Gamma_\mathrm{g}^-$ and $\Gamma_\mathrm{h} = \Gamma_\mathrm{h}^-$ (see Fig. 2). The equations of the boundary value problem are

$$\mathcal{L}u \equiv -\nabla \cdot \boldsymbol{\sigma}^\mathrm{a}(u) = \mathrm{f} \qquad \text{in } \Omega, \tag{26}$$

$$u = \mathrm{g} \qquad \text{on } \Gamma_\mathrm{g}^-, \tag{27}$$

$$\sigma_n^\mathrm{a}(u) = \mathrm{h}^- \quad \text{on } \Gamma_\mathrm{h}^-. \tag{28}$$

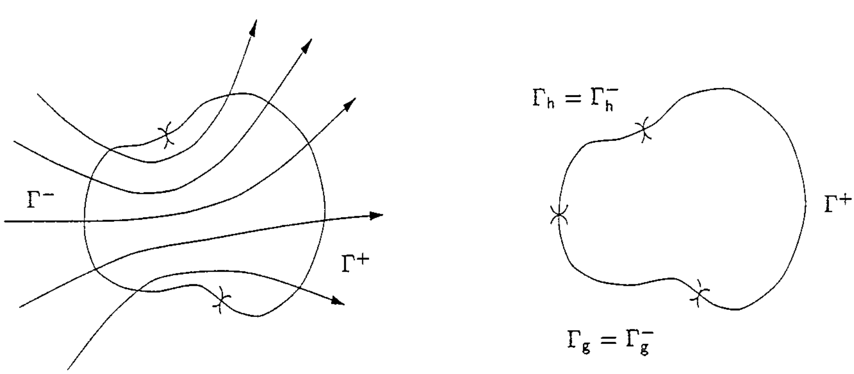



**FIG. 2.** Illustration of boundary partitions for the hyperbolic case.

The bilinear and linear forms are defined as

$$B(w,u) = (\nabla w, \boldsymbol{\sigma}^{\mathrm{a}}(u))_\Omega + (w, a_n^+ u)_\Gamma, \tag{29}$$

$$L(w) = (w, \mathsf{f})_\Omega + (w, \mathsf{h}^-)_{\Gamma_\mathsf{h}^-}. \tag{30}$$

Consistency and stability are established as follows:

*Consistency.*

$$\begin{aligned} 0 &= B(w,u) - L(w) \\ &= -(w, \nabla\cdot\boldsymbol{\sigma}^{\mathrm{a}}(u))_\Omega + (w, -a_n u)_\Gamma + (w, a_n^+ u)_\Gamma - (w,\mathsf{f})_\Omega - (w,\mathsf{h}^-)_{\Gamma_\mathsf{h}^-} \\ &= -(w, \nabla\cdot\boldsymbol{\sigma}^{\mathrm{a}}(u) + \mathsf{f})_\Omega + (w, -a_n^- u - \mathsf{h}^-)_{\Gamma_\mathsf{h}^-} \quad \forall w \in \mathcal{V}. \end{aligned} \tag{31}$$

*Stability.*

$$\begin{aligned} B(w,w) &= (\nabla w, -\boldsymbol{a} w)_\Omega + (w, a_n^+ w)_\Gamma \\ &= -\frac{1}{2}(w, a_n w)_\Gamma + (w, a_n^+ w)_\Gamma \\ &= \frac{1}{2}\left\| |a_n|^{1/2} w \right\|_\Gamma^2 \quad \forall\, w \in \mathcal{V}. \end{aligned} \tag{32}$$

## 2.5. *Finite Element Formulation*

Consider a partition of $\Omega$ into finite elements. Let $\Omega^e$ be the interior of the $e$th element, let $\Gamma^e$ be its boundary, and let

$$\tilde{\Omega} = \bigcup_e \Omega^e \text{ (element interiors)}. \tag{33}$$

Let $\mathcal{S}^h \subset \mathcal{S}$, $\mathcal{V}^h \subset \mathcal{V}$ be *continuous* finite element spaces consisting of polynomials of order $k$ on each element. The classical *continuous Galerkin method* is:

Find $u^h \in \mathcal{S}^h$, such that

$$B(w^h, u^h) = L(w^h) \quad \forall w^h \in \mathcal{V}^h. \tag{34}$$

Stabilized variants are:

*SUPG.*

$$B_{\mathrm{SUPG}}(w^h, u^h) = L_{\mathrm{SUPG}}(w^h), \tag{35}$$

$$B_{\mathrm{SUPG}}(w^h, u^h) \equiv (w^h, u^h) + (\tau \boldsymbol{a}\cdot\nabla w^h, \mathcal{L}u^h)_{\tilde{\Omega}}, \tag{36}$$

$$L_{\mathrm{SUPG}}(w^h) \equiv L(w^h) + (\tau \boldsymbol{a}\cdot\nabla w^h, \mathsf{f})_{\tilde{\Omega}}. \tag{37}$$

*GLS.*

$$B_{\mathrm{GLS}}(w^h, u^h) = L_{\mathrm{GLS}}(w^h), \tag{38}$$

$$B_{\mathrm{GLS}}(w^h, u^h) \equiv B(w^h, u^h) + (\tau \mathcal{L} w^h, \mathcal{L}u^h)_{\tilde{\Omega}}, \tag{39}$$

$$L_{\mathrm{GLS}}(w^h) \equiv L(w^h) + (\tau \mathcal{L} w^h, \mathsf{f})_{\tilde{\Omega}}. \tag{40}$$

*Remarks.*

1. $\tau$ is the stabilization parameter.
2. In the hyperbolic case, or for piecewise linear elements in the elliptic case, SUPG and GLS become identical.
3. Galerkin, SUPG, and GLS are *residual methods*; i.e., Eqs. (34), (35) and (38) are satisfied if $u^h$ is replaced by $u$, the exact solution of the boundary value problem.

## 2.6. *Global Conservation*

To extract the statement of global conservation from the variational formulation, we need to be able to set the weighting function to one. We can only do this if $\Gamma_{\mathsf{g}} = \emptyset$. In this case Eq. (21) yields:

*Elliptic case.*

$$\begin{aligned} 0 &= B(1, u) - L(1) \\ &= \int_\Gamma a_n^+ u \, d\Gamma - \int_\Omega \mathsf{f} \, d\Omega - \int_\Gamma \mathsf{h} \, d\Gamma, \end{aligned} \tag{41}$$

which may be written as

$$0 = \int_{\Gamma^-} \mathsf{h}^- \, d\Gamma + \int_\Omega \mathsf{f} \, d\Omega + \int_{\Gamma^+} (-a_n u + \mathsf{h}^+) \, d\Gamma. \tag{42}$$

*Hyperbolic case.*

$$\begin{aligned} 0 &= B(1, u) - L(1) \\ &= \int_\Gamma a_n^+ u \, d\Gamma - \int_\Omega \mathsf{f} \, d\Omega - \int_{\Gamma_{\mathsf{h}}^-} \mathsf{h}^- \, d\Gamma, \end{aligned} \tag{43}$$

which may be written as

$$0 = \int_{\Gamma^-} \mathsf{h}^- \, d\Gamma + \int_\Omega \mathsf{f} \, d\Omega + \int_{\Gamma^+} (-a_n u) \, d\Gamma. \tag{44}$$

*Remarks.*

1. The *same* conservation results may be derived for the Galerkin finite element method, SUPG, and GLS.
2. Note that for the general case (i.e., $\Gamma_{\mathsf{g}} \neq \emptyset$), *nothing* can be said about global conservation for the finite element methods. However, it is well known how to rectify this situation (see Hughes [6, p. 107]; Hughes *et al*. [7]).

## 3. GLOBAL CONSERVATION FOR THE GENERAL CASE

We assume $\Gamma_{\mathsf{g}} \neq \emptyset$. We shall work with the Galerkin finite element method. We note that the same results can be obtained for SUPG and GLS. Global conservation can be attained by defining an *auxiliary flux* on $\Gamma_{\mathsf{g}}$, denoted $H(\Omega) : \Gamma_{\mathsf{g}} \to \mathbb{R}$, and employing a modified variational formulation. The idea is to add to the weighting function space all the finite element functions associated with $\Gamma_{\mathsf{g}}$. These are omitted in the definition of $\mathcal{V}^h$

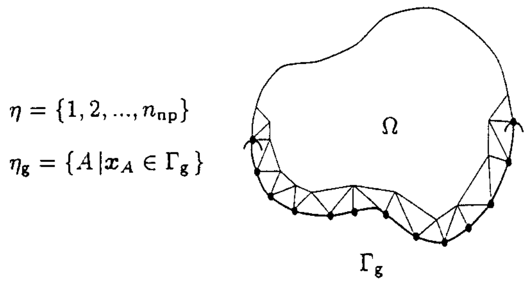



**FIG. 3.** The set of boundary nodes on $\Gamma_g$ and the support of basis functions associated with these nodes.

because functions in $\mathcal{V}^h$ are required to vanish on $\Gamma_g$. Let $\eta$ denote the set of all nodal indices $A = 1, 2, \ldots, n_{np}$. Let $\eta_g$ be the subset corresponding to nodes located on $\Gamma_g$, i.e., $\eta_g = \{A \,|\, \boldsymbol{x}_A \in \Gamma_g\}$ (see Fig. 3).[1] $\mathcal{V}^h$ consists of all functions that are linear combinations of the basis functions associated with nodes $\eta - \eta_g$, viz.,

$$\mathcal{V}^h = \text{span}\{N_A\}_{A \in \eta - \eta_g}, \tag{45}$$

where $N_A$ is the basis function associated with node $\boldsymbol{x}_A$. Let

$$\mathsf{V}^h = \mathcal{V}^h \oplus \text{span}\{N_A\}_{A \in \eta_g}. \tag{46}$$

This is the "completion" of the finite element space. Note that the constant function having value 1 is contained in $\mathsf{V}^h$. The modified form of Galerkin's method is given by:

Find $u^h \in \mathcal{S}^h$ and $H^h(\Omega) \in \mathsf{V}^h - \mathcal{V}^h$ such that

$$(W^h, H^h(\Omega))_{\Gamma_g} = B(W^h, u^h) - L(W^h) \quad \forall\, W^h \in \mathsf{V}^h. \tag{47}$$

Note that (47) splits into two subproblems:

$$0 = B(w^h, u^h) - L(w^h) \qquad \forall\, w^h \in \mathcal{V}^h, \tag{48}$$

$$(W^h, H^h(\Omega))_{\Gamma_g} = B(W^h, u^h) - L(W^h) \quad \forall W^h \in \mathsf{V}^h - \mathcal{V}^h. \tag{49}$$

Note that (48) is the usual problem which defines $u^h \in \mathcal{S}^h$. It is identical to the unmodified case. Equation (49) is a problem which determines $H^h(\Omega)$. In it we assume $u^h$ is already determined by (48), so the right-hand side is completely determined. Furthermore, note that this amounts to a problem involving only nodes on the boundary $\Gamma_g$ and thus may be thought of as a small "postprocessing" calculation. The coefficient matrix for (49) is the "mass matrix" associated with $\Gamma_g$, viz.,

$$\sum_{B \in \eta_g} (N_A, N_B)\, H_B^h(\Omega) = B(N_A, u^h) - L(N_A) \quad \forall\, A \in \eta_g, \tag{50}$$

where $H_B^h(\Omega)$ is the nodal value of $H^h(\Omega)$ at $\boldsymbol{x}_B$. That $H^h(\Omega)$ defines the conserved total flux along $\Gamma_g$ is immediately evident by setting $W^h = 1$ in (47):

[1] When we present schematic diagrams illustrating ideas, for simplicity, we show piecewise linear finite element spaces. However, the results are general and are applicable to spaces of arbitrary order.

*Elliptic case.*

$$\begin{aligned}\int_{\Gamma_{\mathsf{g}}} H^h(\Omega)\, d\Gamma &= (1, H^h(\Omega))_{\Gamma_{\mathsf{g}}} \\ &= B(1, u^h) - L(1) \\ &= \int_{\Gamma_{\mathsf{h}}} a_n^+ u^h \, d\Gamma - \int_\Omega \mathsf{f}\, d\Omega - \int_{\Gamma_{\mathsf{h}}} \mathsf{h}\, d\Gamma \\ &= \int_{\Gamma_{\mathsf{h}}^+} (a_n u^h - \mathsf{h}^+)\, d\Gamma - \int_\Omega \mathsf{f}\, d\Omega - \int_{\Gamma_{\mathsf{h}}^-} \mathsf{h}^-\, d\Gamma, \qquad (51)\end{aligned}$$

or, equivalently,

$$0 = \int_{\Gamma_{\mathsf{g}}} H^h(\Omega)\, d\Gamma + \int_{\Gamma_{\mathsf{h}}^+} (-a_n u^h + \mathsf{h}^+)\, d\Gamma + \int_{\Gamma_{\mathsf{h}}^-} \mathsf{h}^-\, d\Gamma + \int_\Omega \mathsf{f}\, d\Omega. \qquad (52)$$

*Hyperbolic case.*

$$\begin{aligned}\int_{\Gamma_{\mathsf{g}}} H^h(\Omega)\, d\Gamma &= (1, H^h(\Omega))_{\Gamma_{\mathsf{g}}} \\ &= B(1, u^h) - L(1) \\ &= \int_{\Gamma} a_n^+ u^h \, d\Gamma - \int_\Omega \mathsf{f}\, d\Omega - \int_{\Gamma_{\mathsf{h}}^-} \mathsf{h}^-\, d\Gamma \\ &= \int_{\Gamma^+} a_n u^h\, d\Gamma - \int_\Omega \mathsf{f}\, d\Omega - \int_{\Gamma_{\mathsf{h}}^-} \mathsf{h}^-\, d\Gamma, \qquad (53)\end{aligned}$$

or, equivalently,

$$0 = \int_{\Gamma_{\mathsf{g}}^-} H^h(\Omega)\, d\Gamma + \int_{\Gamma^+} (-a_n u^h)\, d\Gamma + \int_{\Gamma_{\mathsf{h}}^-} \mathsf{h}^-\, d\Gamma + \int_\Omega \mathsf{f}\, d\Omega, \qquad (54)$$

where we have used the fact that in this case $\Gamma_{\mathsf{g}} = \Gamma_{\mathsf{g}}^-$.

*Remarks.*

1. Note that in the elliptic case, diffusive flux along $\Gamma_{\mathsf{g}}$ can be computed by adding $a_n u^h = a_n\, \mathsf{g}$ to $H^h$.

2. Boundary fluxes computed in this way exhibit superior convergence behaviour, i.e., “superconvergence,” see Babuška and Miller [1].

## 4. LOCAL CONSERVATION LAWS

The procedure to derive boundary fluxes attaining global conservation, described in the previous section, serves as a paradigm for constructing conserved quantities over subdomains. We shall now start with (47), assuming that $u^h$ and $H^h(\Omega)$ have been obtained, and ask: What is the conserved boundary flux associated with a subdomain consisting of a union of connected elements? Let $\omega \subset \Omega$ denote the subdomain and let $\partial\omega$ denote its boundary. Let $\gamma = \partial\omega - \Gamma$, the part of $\partial\omega$ not contained in $\Gamma$. There are two cases of interest: (i) $\partial\omega \cap \Gamma = \emptyset$ or, at most, consists of a finite number of isolated points in two dimensions,

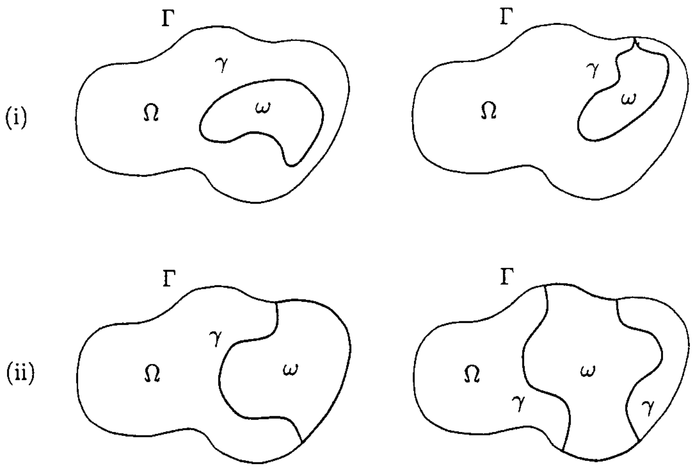



**FIG. 4.** Subdomains consisting of a union of elements. Case (i): $\omega$ is interior to $\Omega$ or intersects $\Gamma$, at most, at a finite number of isolated points in two dimensions, curves in three dimensions, etc. Case (ii): $\omega$ intersects $\Gamma$ and $\partial\omega \cap \Gamma$ is a set of finite measure with respect to the boundary surface form.

curves in three dimensions, etc. (see Fig. 4); and (ii) $\partial\omega \cap \Gamma \neq \emptyset$ and $\partial\omega \cap \Gamma$ has finite measure with respect to the boundary surface form.

We now introduce the field $H^h(\omega)$ defined in terms of the shape functions associated with nodes residing on $\bar{\gamma}$, the closure of $\gamma$. This set of nodes is denoted $\eta_{\bar{\gamma}}$. So

$$H^h(\omega) = \sum_{A \in \eta_{\bar{\gamma}}} N_A \, H^h_A(\omega), \tag{55}$$

and we denote

$$\mathsf{G}^h = \operatorname{span}\{N_A\}_{A \in \eta_{\bar{\gamma}}}. \tag{56}$$

Now our problem is, given $u^h \in \mathcal{S}^h$ and $H^h(\Omega) \in \mathsf{V}^h - \mathcal{V}^h$, the solutions of (48) and (49), respectively, find $H^h(\omega) \in \mathsf{G}^h$ such that

$$(W^h, H^h(\omega))_\gamma = B_\omega(W^h, u^h) - L_\omega(W^h) - (W^h, H^h(\Omega))_{\Gamma_\mathrm{g} \cap \partial\omega} \quad \forall\, W^h \in \mathsf{V}^h, \tag{57}$$

where:

*Elliptic case.*

$$B_\omega(W^h, u^h) \equiv (\nabla W^h, \boldsymbol{\sigma}(u^h))_\omega + (W^h, a_n^+ u^h)_{\Gamma_\mathrm{h} \cap \partial\omega}, \tag{58}$$

$$L_\omega(W^h) \equiv (W^h, \mathsf{f})_\omega + (W^h, \mathsf{h})_{\Gamma_\mathrm{h} \cap \partial\omega}. \tag{59}$$

*Hyperbolic case.*

$$B_\omega(W^h, u^h) \equiv (\nabla W^h, \boldsymbol{\sigma}^\mathrm{a}(u^h))_\omega + (W^h, a_n^+ u^h)_{\Gamma \cap \partial\omega}, \tag{60}$$

$$L_\omega(W^h) \equiv (W^h, \mathsf{f})_\omega + (W^h, \mathsf{h}^-)_{\Gamma_\mathrm{h}^- \cap \partial\omega}. \tag{61}$$

As before, Eq. (57) splits into two problems:

$$(W^h, H^h(\omega))_\gamma = B_\omega(W^h, u^h) - L_\omega(W^h) - (W^h, H^h(\Omega))_{\Gamma_{\mathsf{g}} \cap \partial\omega} \quad \forall\, W^h \in \mathsf{G}^h, \tag{62}$$

and

$$0 = B_\omega(W^h, u^h) - L_\omega(W^h) - (W^h, H^h(\Omega))_{\Gamma_{\mathsf{g}} \cap \partial\omega} \quad \forall\, W^h \in \mathsf{V}^h - \mathsf{G}^h. \tag{63}$$

The matrix counterpart of the first problem serves to define $H^h(\omega)$:

$$\sum_{B \in \eta_{\bar{\gamma}}} (N_A, N_B)_\gamma H_B^h(\omega) = B_\omega(N_A, u^h) - L_\omega(N_A) - (N_A, H^h(\Omega))_{\Gamma_{\mathsf{g}} \cap \partial\omega} \quad \forall A \in \eta_{\bar{\gamma}}. \tag{64}$$

The second problem is an identity by virtue of Eq. (47). To see this, select a $W^h$ whose support is contained entirely within $\omega$. In this case

$$B(W^h, u^h) = B_\omega(W^h, u^h), \tag{65}$$

$$L(W^h) = L_\omega(W^h), \tag{66}$$

$$(W^h, H^h(\Omega))_{\Gamma_{\mathsf{g}}} = (W^h, H^h(\Omega))_{\Gamma_{\mathsf{g}} \cap \partial\omega}. \tag{67}$$

Consequently, (63) follows from (47) in this case. For $W^h$ having support entirely outside of $\omega$, all terms in (63) are identically zero.

The conservation laws implied by (57) are established by selecting any $W^h \in \mathsf{V}^h$ such that

$$W^h|_\omega = 1. \tag{68}$$

With this selection, we have:

*Elliptic case.*

$$\begin{aligned} \int_\gamma H^h(\omega)\, d\gamma &= B_\omega(1, u^h) - L_\omega(1) - \int_{\Gamma_{\mathsf{g}} \cap \partial\omega} H^h(\Omega)\, d\Gamma \\ &= \int_{\Gamma_{\mathsf{h}} \cap \partial\omega} a_n^+ u^h\, d\Gamma - \int_\omega \mathsf{f}\, d\omega - \int_{\Gamma_{\mathsf{h}} \cap \partial\omega} \mathsf{h}\, d\Gamma - \int_{\Gamma_{\mathsf{g}} \cap \partial\omega} H^h(\Omega)\, d\Gamma \\ &= \int_{\Gamma_{\mathsf{h}}^+ \cap \partial\omega} (a_n^+ u^h - \mathsf{h}^+)\, d\Gamma - \int_\omega \mathsf{f}\, d\omega \\ &\quad - \int_{\Gamma_{\mathsf{h}}^- \cap \partial\omega} \mathsf{h}^-\, d\Gamma - \int_{\Gamma_{\mathsf{g}} \cap \partial\omega} H^h(\Omega)\, d\Gamma, \end{aligned} \tag{69}$$

or, equivalently,

$$\begin{aligned} 0 = \int_\gamma H^h(\omega)\, d\gamma &+ \int_{\Gamma_{\mathsf{h}}^+ \cap \partial\omega} (-a_n^+ u^h + \mathsf{h}^+)\, d\Gamma \\ &+ \int_{\Gamma_{\mathsf{h}}^- \cap \partial\omega} \mathsf{h}^-\, d\Gamma + \int_{\Gamma_{\mathsf{g}} \cap \partial\omega} H^h(\Omega)\, d\Gamma + \int_\omega \mathsf{f}\, d\omega. \end{aligned} \tag{70}$$

*Hyperbolic case.*

$$\begin{aligned}
\int_\gamma H^h(\omega)\,d\gamma &= B_\omega(1,u^h) - L_\omega(1) - \int_{\Gamma_\mathsf{g}\cap\partial\omega} H^h(\Omega)\,d\Gamma \\
&= \int_{\Gamma\cap\partial\omega} a_n^+ u^h\,d\Gamma - \int_\omega \mathsf{f}\,d\omega - \int_{\Gamma_\mathsf{h}^-\cap\partial\omega} \mathsf{h}^-\,d\Gamma - \int_{\Gamma_\mathsf{g}\cap\partial\omega} H^h(\Omega)\,d\Gamma \\
&= \int_{\Gamma^+\cap\partial\omega} a_n u^h\,d\Gamma - \int_\omega \mathsf{f}\,d\omega - \int_{\Gamma_\mathsf{h}^-\cap\partial\omega} \mathsf{h}^-\,d\Gamma - \int_{\Gamma_\mathsf{g}\cap\partial\omega} H^h(\Omega)\,d\Gamma, \qquad (71)
\end{aligned}$$

or, equivalently,

$$0 = \int_\gamma H^h(\omega)\,d\gamma + \int_{\Gamma^+\cap\partial\omega} (-a_n u^h)\,d\Gamma + \int_{\Gamma_\mathsf{h}^-\cap\partial\omega} \mathsf{h}^-\,d\Gamma + \int_{\Gamma_\mathsf{g}^-\cap\partial\omega} H^h(\Omega)\,d\Gamma + \int_\omega \mathsf{f}\,d\omega, \qquad (72)$$

where, again, we have used the fact that $\Gamma_\mathsf{g} = \Gamma_\mathsf{g}^-$ in the hyperbolic case.

*Uniqueness.* We might ask the following question: Suppose we performed a similar construction for the complementary subdomain $\Omega - \omega$. What is the relationship between $H^h(\Omega-\omega)$ and $H^h(\omega)$? We obviously would hope that they would be the same up to a sign reversal. A simple argument verifies this.

By analogy with Eq. (57), we have

$$\begin{aligned}
&(W^h, H^h(\Omega-\omega))_\gamma \\
&\quad = B_{\Omega-\omega}(W^h,u^h) - L_{\Omega-\omega}(W^h) - (W^h, H^h(\Omega))_{\Gamma_\mathsf{g}\cap\partial(\Omega-\omega)} \quad \forall W^h \in \mathsf{V}^h. \qquad (73)
\end{aligned}$$

Add (57) and (73),

$$\begin{aligned}
(W^h, H^h(\omega) + H^h(\Omega-\omega))_\gamma &= B_\omega(W^h,u^h) + B_{\Omega-\omega}(W^h,u^h) - L_\omega(W^h) \\
&\quad - L_{\Omega-\omega}(W^h) - (W^h, H^h(\Omega))_{\Gamma_\mathsf{g}\cap\partial\omega} \\
&\quad - (W^h, H^h(\Omega))_{\Gamma_\mathsf{g}\cap\partial(\Omega-\omega)} \\
&= B(W^h,u^h) - L(W^h) - (W^h, H^h(\Omega))_{\Gamma_\mathsf{g}} \\
&= 0 \quad \forall\, W^h \in \mathsf{V}^h \qquad (74)
\end{aligned}$$

by (47). Now restrict $W^h$ to $\mathsf{G}^h \subset \mathsf{V}^h$,

$$(W^h, H^h(\omega) + H^h(\Omega-\omega))_\gamma = 0 \quad \forall\, W^h \in \mathsf{G}^h, \qquad (75)$$

which is equivalent to the matrix problem

$$\sum_{B\in\eta_{\bar\gamma}} (N_A, N_B)_\gamma \left(H_B^h(\omega) + H_B^h(\Omega-\omega)\right) = 0 \quad \forall A \in \eta_{\bar\gamma}, \qquad (76)$$

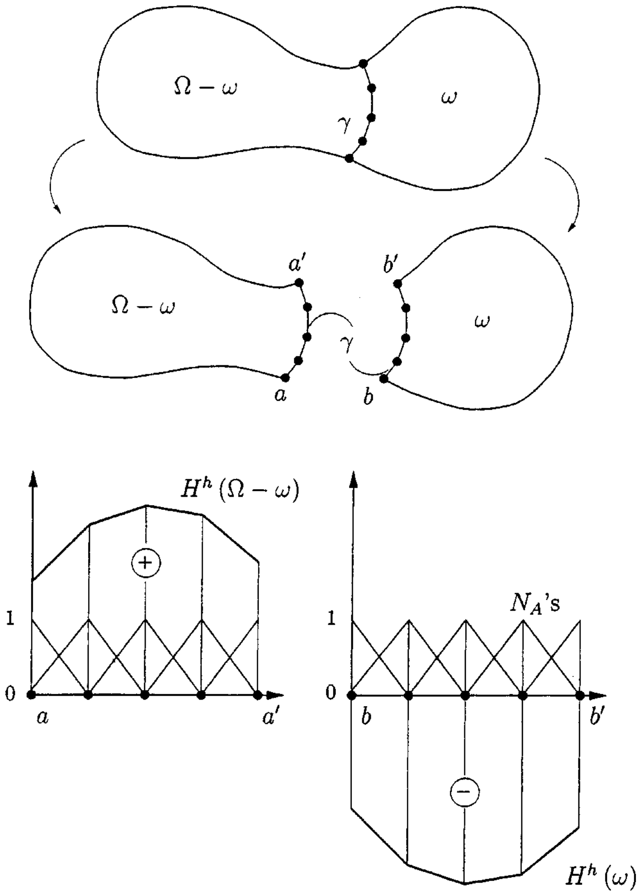



**FIG. 5.** Subdomain interface fluxes $H^h(\omega)$ and $H^h(\Omega - \omega)$ equilibrate pointwise and are conservative with respect to subdomains $\omega$ and $\Omega - \omega$, respectively.

from which it follows that, pointwise on $\gamma$,

$$H^h(\omega) \equiv -H^h(\Omega - \omega). \tag{77}$$

See Fig. 5 for a schematic illustration of this result.

## 5. ELEMENT CONSERVATION LAWS

The results obtained in Section 3 for an arbitrary subset of connected elements can be specialized to an individual element. Simply set $\omega = \Omega^e$, for $e$ fixed. As before, let $\gamma = \gamma^e \equiv \Gamma^e - \Gamma$, where $\Gamma^e = \partial\Omega^e$. Now (57) becomes

$$(W^h, H^h(\Omega^e))_{\gamma^e} = B_{\Omega^e}(W^h, u^h) - L_{\Omega^e}(W^h) - (W^h, H^h(\Omega))_{\Gamma_g \cap \Gamma^e} \quad \forall\, W^h \in \mathsf{V}^h, \tag{78}$$

where:

*Elliptic case.*

$$B_{\Omega^e}(W^h, u^h) \equiv (\boldsymbol{\nabla} W^h, \boldsymbol{\sigma}(u^h))_{\Omega^e} + (W^h, a_n^+ u^h)_{\Gamma_{\mathsf{h}}\cap\Gamma^e}, \tag{79}$$

$$L_{\Omega^e}(W^h) \equiv (W^h, \mathsf{f})_{\Omega^e} + (W^h, \mathsf{h})_{\Gamma_{\mathsf{h}}\cap\Gamma^e}. \tag{80}$$

*Hyperbolic case.*

$$B_{\Omega^e}(W^h, u^h) \equiv (\boldsymbol{\nabla} W^h, \boldsymbol{\sigma}^{\mathsf{a}}(u^h))_{\Omega^e} + (W^h, a_n^+ u^h)_{\Gamma\cap\Gamma^e}, \tag{81}$$

$$L_{\Omega^e}(W^h) \equiv (W^h, \mathsf{f})_{\Omega^e} + (W^h, \mathsf{h}^-)_{\Gamma_{\mathsf{h}}^-\cap\Gamma^e}. \tag{82}$$

Let $\eta^e$ denote the node numbers of nodes attached to $\bar{\gamma}^e$. Then (78) reduces to the local problem

$$\begin{aligned}\sum_{B\in\eta^e} (N_A, N_B)_{\gamma^e} H_B^h(\Omega^e) = \mathsf{f}_A^e &\equiv B_{\Omega^e}(N_A, u^h) - L_{\Omega^e}(N_A) - (N_A, H^h(\Omega))_{\Gamma_{\mathsf{g}}\cap\Gamma^e} \\ &= \int_{\Omega^e} \boldsymbol{\nabla} N_A \cdot \boldsymbol{\sigma}(u^h)\, d\Omega - \int_{\Gamma_{\mathsf{h}}^+\cap\Gamma^e} N_A(-a_n^+ u^h + \mathsf{h}^+)\, d\Gamma \\ &\quad - \int_{\Gamma_{\mathsf{h}}^-\cap\Gamma^e} N_A \mathsf{h}^-\, d\Gamma - \int_{\Gamma_{\mathsf{g}}\cap\Gamma^e} N_A\, H^h(\Omega)\, d\Gamma \\ &\quad - \int_{\Omega^e} N_A\, \mathsf{f}\, d\Omega \quad \forall A \in \eta^e. \end{aligned} \tag{83}$$

We refer to $\mathsf{f}_A^e$ as the *e*th element contribution to the flux at node $A$, or simply, the *element nodal flux*. From Eq. (78), we have the element conservation laws:

*Elliptic case.*

$$\begin{aligned} 0 = &\int_{\gamma^e} H^h(\Omega^e)\, d\Gamma + \int_{\Gamma_{\mathsf{h}}^+\cap\Gamma^e} (-a_n^+ u^h + \mathsf{h}^+)\, d\Gamma \\ &+ \int_{\Gamma_{\mathsf{h}}^-\cap\Gamma^e} \mathsf{h}^-\, d\Gamma + \int_{\Gamma_{\mathsf{g}}\cap\Gamma^e} H^h(\Omega)\, d\Gamma + \int_{\Omega^e} \mathsf{f}\, d\Omega. \end{aligned} \tag{84}$$

*Hyperbolic case.*

$$\begin{aligned} 0 = &\int_{\gamma^e} H^h(\Omega^e)\, d\Gamma + \int_{\Gamma^+\cap\Gamma^e} (-a_n^+ u^h)\, d\Gamma \\ &+ \int_{\Gamma_{\mathsf{h}}^-\cap\Gamma^e} \mathsf{h}^-\, d\Gamma + \int_{\Gamma_{\mathsf{g}}^-\cap\Gamma^e} H^h(\Omega)\, d\Gamma + \int_{\Omega^e} \mathsf{f}\, d\Omega. \end{aligned} \tag{85}$$

We also note that from the uniqueness argument presented in the preceeding section, we have the pointwise conservation relationship

$$H^h(\Omega^e) = -H^h(\Omega - \Omega^e). \tag{86}$$

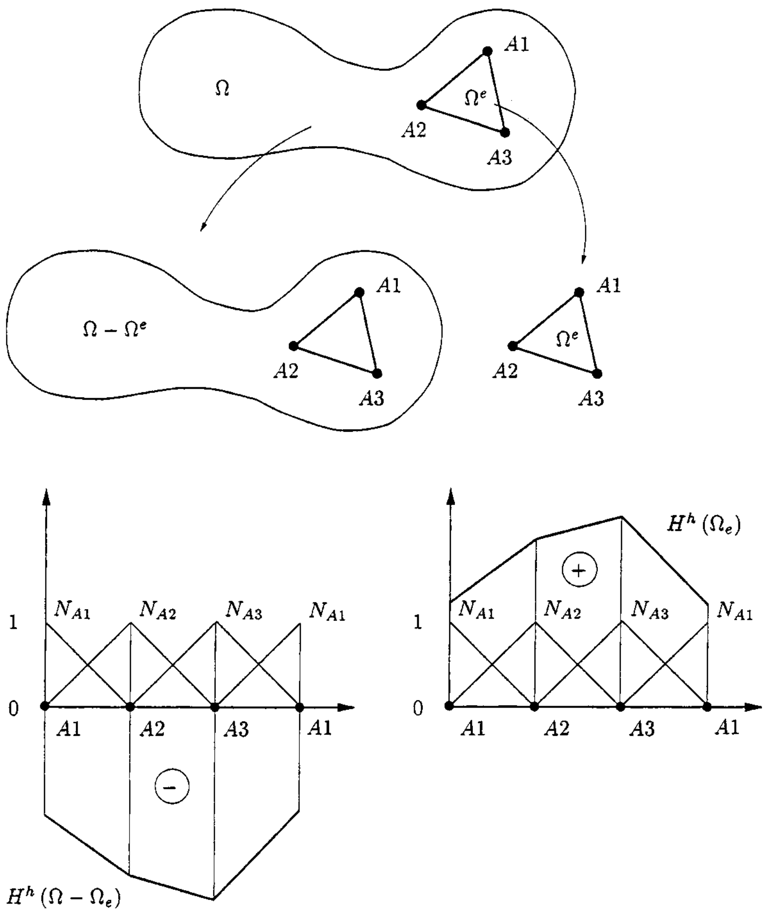



**FIG. 6.** Element $\Omega^e$ interface flux $H^h(\Omega^e)$ equilibrates subdomain $\Omega$–$\Omega^e$ interface flux $H^h(\Omega$–$\Omega^e)$ pointwise. $H^h(\Omega^e)$ and $H^h(\Omega$–$\Omega^e)$ are conservative with respect to $\Omega^e$ and $\Omega$–$\Omega^e$, respectively.

See Fig. 6. By summing Eq. (83) over $A \in \eta^e$, we see that

$$\int_{\gamma^e} H^h(\Omega^e)\, d\Gamma = \sum_{A \in \eta^e} \mathrm{f}_A^e. \tag{87}$$

Thus, by the element conservation laws, (84) and (85), we see that the sum of the element nodal fluxes represents a conserved quantity. See Figs. 7 and 8 for schematic illustrations of the element conservation laws.

By returning to the global equation (47) and selecting $W^h = N_A$, for $A$ fixed, we see that

$$\begin{aligned} 0 &= B(N_A, u^h) - L(N_A) - (N_A, H^h(\Omega))_{\Gamma_\mathrm{g}} \\ &= \sum_{e \in E(A)} \left(B_{\Omega^e}(N_A, u^h) - L_{\Omega^e}(N_A) - (N_A, H^h(\Omega))_{\Gamma_\mathrm{g} \cap \Gamma^e}\right) \\ &= \sum_{e \in E(A)} \mathrm{f}_A^e, \end{aligned} \tag{88}$$

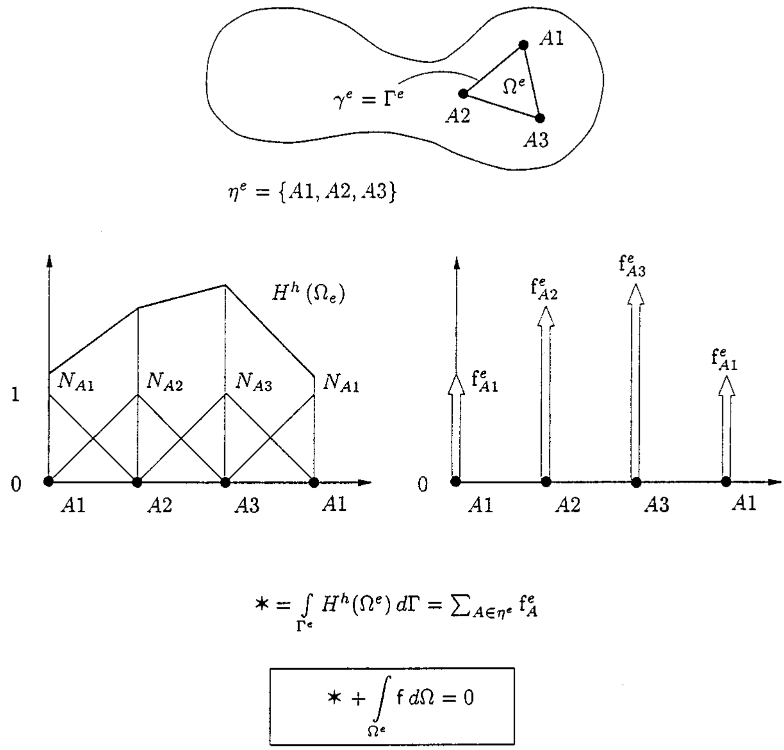



**FIG. 7.** Element conservation law ($\text{meas}(\Gamma^e \cap \Gamma) = 0$). $H^h(\Omega^e)$ is the conservative redistribution of the nodal fluxes, $\mathrm{f}^e_A$, in terms of the basis functions, $N_A$.

where $E(A)$ is the set of element numbers of elements attached to node $A$. See Fig. 9. The $\mathrm{f}^e_A$'s may be thought of as a delta distribution representation of the conserved fluxes.

## 6. NUMERICAL EXAMPLES: CONSERVATIVE FLUX CALCULATION

We consider the following boundary value problem: Find $u$ such that

$$-\Delta u = 1 \quad \text{in } \Omega, \tag{89}$$

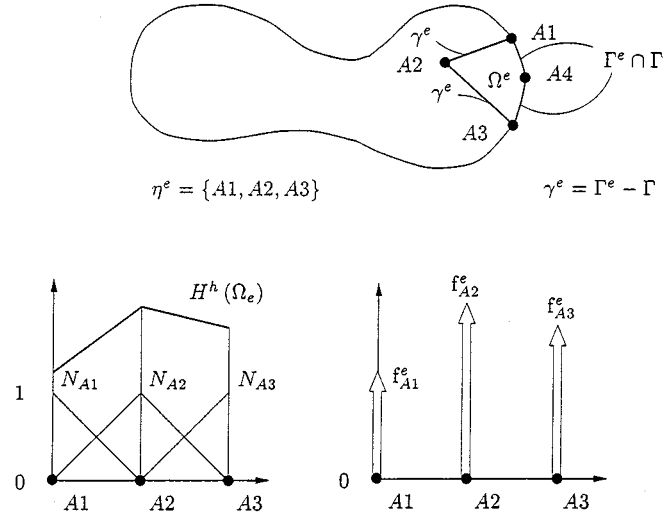
$$u = 0 \quad \text{on } \Gamma, \tag{90}$$

where $\Omega = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = [-1, 0.5] \times [-1, 1]$ and $\Omega_2 = [0.5, 1] \times [-1, 1]$, with boundaries $\Gamma_1$ and $\Gamma_2$, respectively. An approximate solution is computed using the standard continuous Galerkin method with piecewise linears on unstructured triangulations which respect the subdomains $\Omega_1$ and $\Omega_2$, see Fig. 10.

We calculate approximations of the normal flux

$$\sigma_n = \boldsymbol{n} \cdot \nabla u \tag{91}$$

on the boundaries $\Gamma$, $\Gamma_1$, and $\Gamma_2$ using the methodology described previously. We shall refer to approximations computed in this manner as the "conservative flux." For comparison, we also calculate the exact flux using a Fourier series solution and a numerical approximation

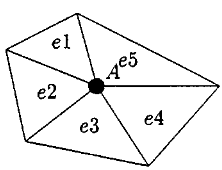



$$\ast = \int_{\Gamma^e} H^h(\Omega^e)\, d\Gamma = \textstyle\sum_{A\in\eta^e} f^e_A$$

$$\ast + \int_{\Gamma_h^+\cap\Gamma^e} \left(-a_n^+ u^h + h^+\right) d\Gamma + \int_{\Gamma_h^-\cap\Gamma^e} h^- \, d\Gamma + \int_{\Gamma_g\cap\Gamma^e} H^h(\Omega)\, d\Gamma + \int_{\Omega^e} f \, d\Omega = 0$$

**FIG. 8.** Element conservation law (meas($\Gamma^e \cap \Gamma$) $\neq 0$).

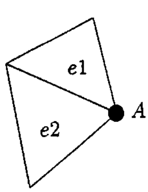
Internal nodes




$E(A) = \{e1, e2, e3, e4, e5\}$

$$f^e_A = \int_{\Omega^e} \nabla N_A \cdot \sigma\left(u^h\right) d\Omega - \int_{\Omega^e} N_A f \, d\Omega$$

Boundary nodes




$E(A) = \{e1, e2\}$

$$f^e_A = \int_{\Omega^e} \nabla N_A \cdot \sigma\left(u^h\right) d\Omega - \int_{\Omega^e} N_A f \, d\Omega - \int_{\Gamma_h\cap\Gamma^e} N_A\left(-a_n^+ u^h + h\right) d\Gamma - \int_{\Gamma_g\cap\Gamma^e} N_A H^h(\Omega)\, d\Gamma$$

$$0 = \sum_{e\in E(A)} f^e_A$$

**FIG. 9.** Conservation of nodal fluxes.

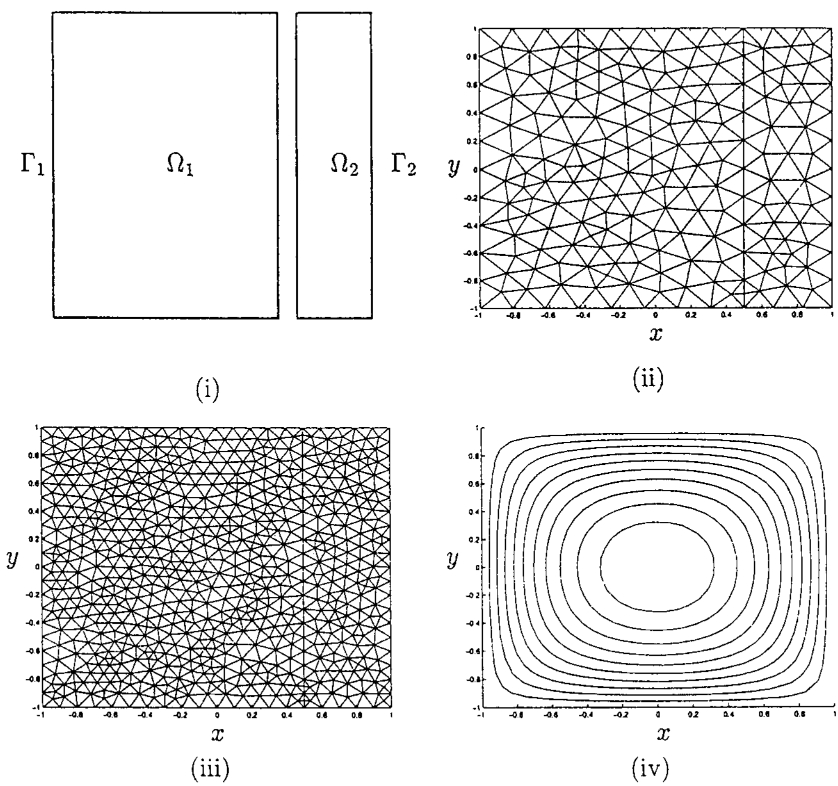



**FIG. 10.** (i) Domain $\Omega = \Omega_1 \cup \Omega_2$. (ii) Coarse mesh with 358 triangles and 201 nodes. (iii) Fine mesh with 1342 triangles and 712 nodes. (iv) Contours of the numerical solution on the fine mesh.

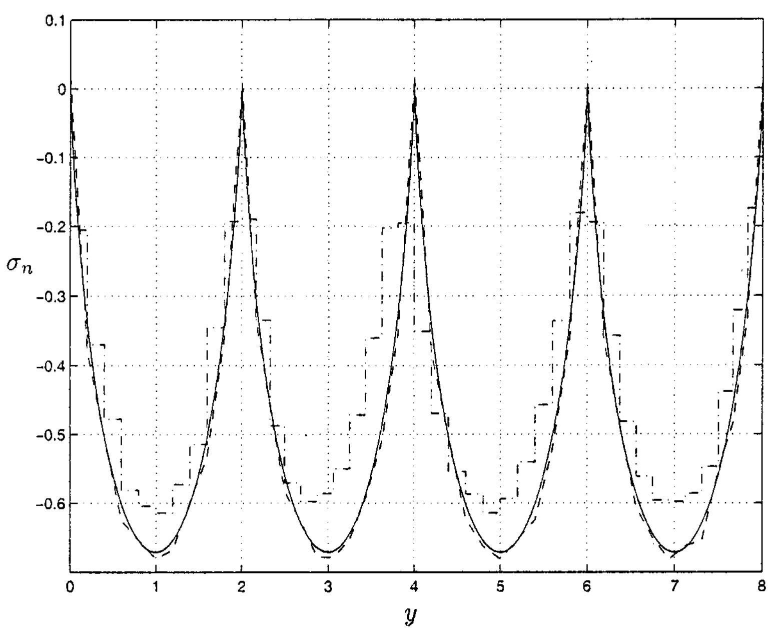



**FIG. 11.** The exact flux (solid), the conservative flux (dashed), and the flux computed by direct evaluation of element derivatives (dash-dotted) are plotted on the boundary starting in the upper left-hand corner $(-1, 1)$ in counter-clockwise fashion.

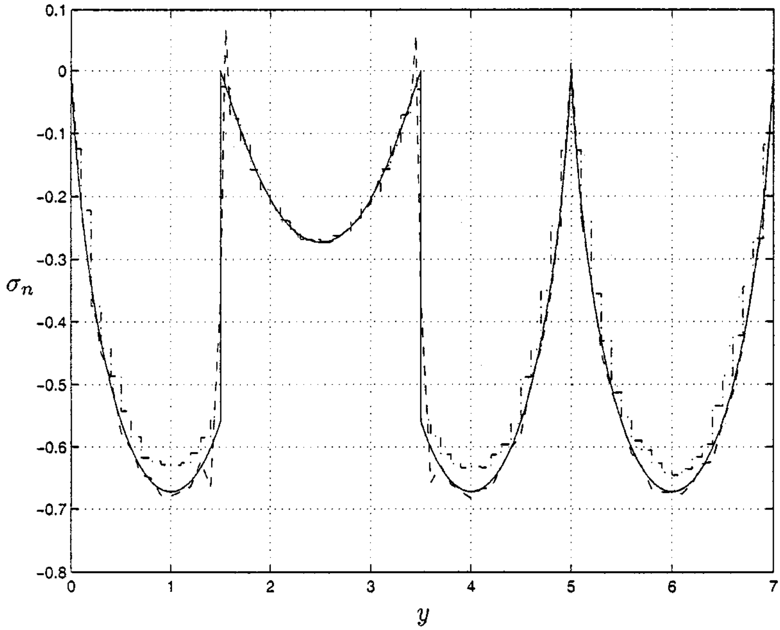



**FIG. 12.** The exact flux (solid), the conservative flux (dashed), and the directly evaluated flux (dash-dotted) plotted as functions on $\Gamma_1$ starting in the corner $(-1, -1)$ in counter-clockwise direction.

obtained by direct evaluation of the flux, obtained by computing derivatives of the numerical solution in elements adjacent to the boundary in question.

In Fig. 11 we present fluxes for the external boundary $\Gamma$. The approximations to the exact solution were computed using the coarse mesh. We observe that the conservative flux faithfully approximates the exact solution, whereas the direct evaluation of flux is significantly in error. Furthermore, the conservative flux is verified to satisfy the conservation law, namely,

$$\int_{\Omega} \mathsf{f} + \int_{\Gamma} \sigma_n = 0, \tag{92}$$

to machine precision, as anticipated by the theory.

In Figs. 12 and 13, we show the fluxes on $\Gamma_1$ and $\Gamma_2$, respectively. The numerical calulations were performed on the fine mesh. In the case of the evaluation of flux by direct calculation of element derivatives, we have computed the flux on the internal interface as the average of the fluxes computed on elements to the left and right of the interface. This substantially improves these results, as can be seen by comparing Fig. 14 with Figs. 12 and 13. We note, that for the calculation of the conservative flux we again satisfy the conservation law

$$\int_{\Omega_i} \mathsf{f} + \int_{\Gamma_i} \sigma_n = 0 \quad \text{for } i = 1, 2, \tag{93}$$

to machine precision, consistent with the theory. Further, the conservative flux is a much more accurate approximation than the direct evaluation of flux. However, on the interface region, the averaged direct evaluation produces commensurate accuracy, but does not attain

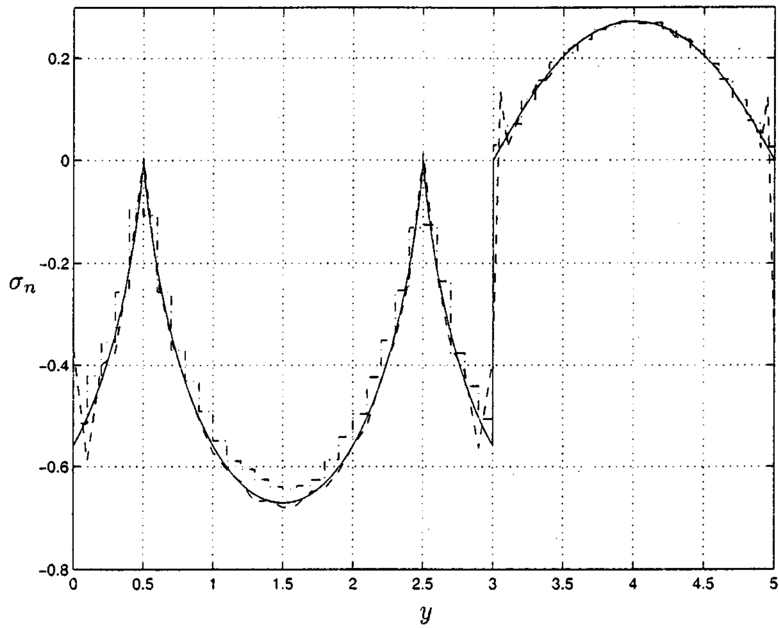



**FIG. 13.** The exact flux (solid), the conservative flux (dashed), and the directly calculated flux (dash-dotted) as functions on $\Gamma_2$ starting in $(0.5, -1)$ in counter-clockwise direction.

conservation. The only negative aspect of the conservative flux calculation is that we have enforced continuity of flux around the endpoints of the interface where the exact solution is discontinuous. As might be anticipated by virtue of the fact that the conservative flux calculation amounts to an $L_2$-projection, overshoots and undershoots are exhibited at points

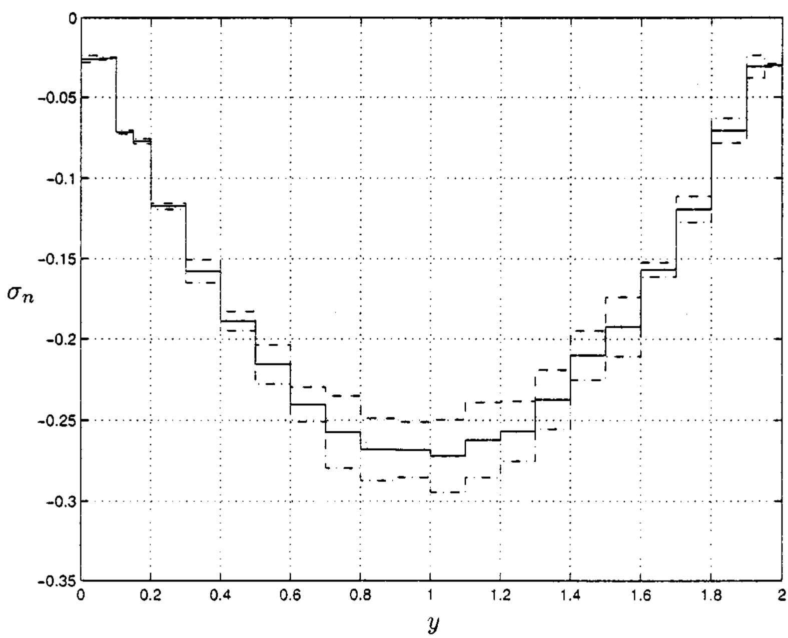



**FIG. 14.** Comparison between: left (dashed), right (dash-dotted), and average (solid) of the directly evaluated fluxes on the internal interface.

of discontinuity of the exact solution. This result suggests that the conservative flux approximation should be allowed to be discontinuous at known locations of discontinuity in the exact solution.

## 7. CONCLUSIONS

1. From (88) we see that the element nodal fluxes are *conserved node-wise*.

2. Likewise, an individual element's nodal fluxes are a *conserved* quantity by virtue of (87).

3. The $H^h(\Omega^e)$ represents a continuous redistribution of element $e$'s nodal fluxes, in terms of the basis functions, that *preserves conservation*.

4. For the stabilized methods, the element fluxes change, but the conservation laws remain the same.

5. Nodal fluxes correspond to the notion of nodal forces in structural mechanics. Structural engineers seem comfortable with element nodal force resultants (i.e., the $\mathrm{f}_A^e$'s here) whereas fluid mechanicians do not. Rather, fluid mechanicians seem comfortable with distributed fluxes over the boundaries of control volumes (e.g., element domains). We see from the preceding developments that nodal fluxes and their continuous redistribution in terms of the element basis functions are different but equivalent representations of the same information, viz.,

$$\sum_{B\in\eta^e} (N_A, N_B)_{\gamma^e} H_B^h(\Omega^e) = \mathrm{f}_A^e \quad \forall A \in \eta^e, \tag{94}$$

$$\mathrm{f}_A^e = (N_A, H^h(\Omega^e))_{\gamma^e} \quad \forall A \in \eta^e. \tag{95}$$

If we know the $\mathrm{f}_A^e$'s, $H^h(\Omega^e)$ is uniquely defined by (94). Likewise, if we know $H^h(\Omega^e)$, the $\mathrm{f}_A^e$'s are uniquely defined by (95). These quantities are fundamental to the local conservation structure of continuous Galerkin and stabilized methods.

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